JOURNAL OF DIFFERENTIAL EQUATIONS 68, 10–21 (1987)

Generalized Viscosity Solutions for Hamilton–Jacobi Equations with Time-Measurable Hamiltonians

EMMANUEL NICHOLAS BARRON*

AND

ROBERT JENSEN†

Department of Mathematical Sciences, Loyola University of Chicago, Chicago, Illinois 60626

Received January 27, 1986; revised May 12, 1986

INTRODUCTION

The objective of this paper is to extend the idea of viscosity solution for nonlinear first order Hamilton–Jacobi equations, $u_t + H(t, x, D_x u) = 0$, with time-continuous H to time-measurable hamiltonians. We are primarily motivated by the fact that control problems, for which the value function is the viscosity solution of a Hamilton–Jacobi equation, should not be restricted to continuity in time. In Barron and Jensen [1] we were confronted with a linear Hamilton–Jacobi equation arising in the proof of the Pontryagin which had time-measurable coefficients. The existing theory of viscosity solutions did not apply to even this case. Moreover, important models of controlled first order Hamilton–Jacobi equations require time-measurable controls. For example, models of controlled traffic flow or queueing processes might be of this type.

We are about then to extend the definition of viscosity solution to "generalized viscosity solution" applying to time-measurable hamiltonians. In this paper we give the definition and derive the corresponding uniqueness result with an implicit domain of dependence consequent. The fundamental technique used in the proof is the so-called "blow up" method which has also been used elsewhere [5], although not in the generality discussed here. This method effectively freezes the point (t_0, x_0) under consideration and essentially removes it from the problem. Of course t_0 is

* Partially supported by a grant from the AFOSR, No. AFOSR-86-0202.

† Alfred P. Sloan Foundation Fellow. Also partially supported by a grant from the National Science Foundation.

0022-0396/87 $3.00

causing the problems. With this point frozen standard viscosity solutions can be used. Therefore the blow up method itself is of substantial independent interest.

Time-measurable hamiltonians were first considered by Ishii [7] by a method completely different from ours. Ishii's method essentially involves approximation by continuous hamiltonians. The method we use instead involves restricting the class of test functions used in the Crandall, Evans, and Lions [4] definition of viscosity solution. We believe that this method has some advantages. First, it seems possible to extend our method to hamiltonians which are also only measurable in x since we freeze both t and x using the blow ups. So both t and x are eliminated from the problem. Second, our method does not involve the choice or establishment of any continuous approximation to the hamiltonian. Our definition looks at the hamiltonian at Lebesgue points and so, in a vague sense is a natural way to view the problem. In any case we do present another way of looking at such problems which also holds promise for the second order equations which R. Jensen is considering.

Ishii establishes existence and uniqueness results under more general hypotheses than assumed here. We believe that our method extends to Ishii's hypotheses but we leave this as an open problem. We also believe that our definition of viscosity solution for time-measurable hamiltonians is equivalent to Ishii's but this is also open.

This paper is organized as follows. In Section 1 we make the definition of generalized viscosity solution. We require that the solution be Lipschitz and that $H(\cdot, \cdot, p)$ is Lipschitz in p. Since our motivation is control problems this is a reasonable hypothesis. However, it is much more restrictive than Ishii's conditions and the extension of our result to Ishii's hypotheses is open. On the other hand, proving that our formulation extended to continuous viscosity solutions is equivalent to Ishii's will do the job in one stroke.

Section 2 is the discussion of the blow-up method and is fundamental to this paper. Section 3 contains our proof of a pointwise comparison principle (Theorem 1.1) from which uniqueness follows immediately.

Lions and Perthame [8] have also considered time measurable hamiltonians. Their approach uses accretive operator ideas. The main result of [8] is that the Lions–Perthame definition and the definition of Ishii are equivalent.

Finally, let us mention that the results of this paper were obtained before we were aware of either Ishii's or Lions–Perthame's results. Their existence was made known to us by Crandall to whom we are grateful.

1. Generalized Viscosity Solutions

In this paper we are primarily concerned with the Hamilton–Jacobi equation which we shall, for simplicity in what follows, write as follows

$$\partial u/\partial t + H(t, x, D_x u) = f(t, x) \tag{1.1}$$

for (t, x) in $[0, T) \times \Omega$, where Ω is an open set in R^n. We will have a terminal condition associated with Eq. (1.1) since we have in mind control problems as examples:

$$u(T, x) = g(x), \qquad x \in \Omega. \tag{1.2}$$

For the hamiltonian H: $[0, T] \times \Omega \times R^n \to R^1$ and the functions f and g we assume

(H) $H(t, x, p)$ is Lebesgue measurable in t and uniformly continuous in (x, p) on compact subsets of $[0, T] \times \Omega \times R^n$. The function f: $[0, T] \times \Omega \to R^1$ is assumed to be Lebesgue measurable in t and uniformly continuous in x on compact subsets of $[0, T] \times \Omega$. The function g: $\Omega \to R^1$ is assumed uniformly continuous on compact subsets of Ω. Further $H(t, \cdot, \cdot)$ and $f(t, \cdot)$ are in $L^1(0, T)$ and there is $L > 0$ so that $|H(t, x, p) - H(t, x, q)| \leqslant L|p - q|$.

We require the following for use in our definition of generalized viscosity solution.

Definition. Let F: $\Omega \subset R^n \to R^1$ be Lebesgue measurable. A set $Q \subset \Omega$ is said to be a *uniform Lebesgue set of F* if

$$\lim_{r \to 0} \sup_{z \in Q} \frac{1}{m(B_r)} \int_{B_r} |F(z + y) - F(z)| \, dy = 0,$$

where m is n-dimensional Lebesgue measure and B_r is the ball of radius r centered at the origin.

Remark. Since we are extending the theory of viscosity solutions from continuous hamiltonians (in t) to measurable hamiltonians in t, it is natural to consider Lebesgue points. The preceding definition will play a central role.

We now can define our notion of generalized viscosity solution:

Definition. A bounded Lipschitz continuous function u: $[0, T] \times \Omega \to R^1$ is a *generalized viscosity supersolution* (resp., *subsolution*) of (1.1) if for any sequence of points $\{(t_i, x_i)\}$ in $[0, T] \times \Omega$ and sequence of functions $\{\phi_i\}$, with $\phi_i \in C^2([0, T] \times \Omega)$ satisfying

(i) $(t_i, x_i) \to (t_0, x_0)$ as $i \to \infty$;

(ii) for any compact subsets X and P of Ω, the set $\{t_0\} \times X \times P$ is a uniform Lebesgue set of H and $\{t_0\} \times X$ is a uniform Lebesgue set of f;

(iii) there is a constant $k > 0$ such that, for each i, $\sup\{|D^2_{x,t}\phi_i(t, x)|;\ (t, x) \in [0, T] \times \Omega\} \leqslant k/|t_i - t_0|$ (where $D^2_{x,t}$ means any second order derivative);

(iv) for each i, $u - \phi_i$ has a minimum (resp. maximum) of 0 attained at (t_i, x_i).

Then

$$\limsup_{i \to \infty} [\partial\phi_i(t_i, x_i)/\partial t + H(t_0, x_0, D_x\phi_i(t_i, x_i))] \leqslant f(t_0, x_0)$$

(resp.

$$\liminf_{i \to \infty} [\partial\phi_i(t_i, x_i)/\partial t + H(t_0, x_0, D_x\phi_i(t_i, x_i))] \geqslant f(t_0, x_0)).$$

A Lipschitz function which is both a generalized viscosity sub and supersolution is a generalized viscosity solution of (1.1) and (1.2) if $u(T, x) = g(x)$, $x \in \Omega$.

Remark. Condition (iii) is critical and is a funcamental difference in the ordinary definition of viscosity solution. We are putting a bound on the rate at which the second derivatives explode as $t_i \to t_0$, independent of x. Also the condition (iii) cannot be expressed in terms of $D\overset{+}{u}$, $D\bar{u}$ and so a basic difference in approaches is also evident.

Remark. Lemma 2.2 will establish conditions under which (ii) holds.

Remark. Our definition of generalized viscosity solution is completely different from that of Lions [8] or Ishii [7]. Lions is motivated by accretive operator methods and Ishii by the approach of (essentially) approximating by continuous hamiltonians.

The primary goal of this paper is to prove.

THEOREM 1.1. *Let (H) hold for two functions $f_1(t, x)$ and $f_2(t, x)$. Let u_1 (resp., u_2) be a generalized viscosity subsolution (resp. supersolution) of (1.1) with f replaced by f_1 (resp. f_2). Then, we have at each $(t_0, x_0) \in [0, T) \times \Omega$,*

$$u_1(t_0, x_0) - u_2(t_0, x_0) \leqslant \sup_{x \in \Omega} (u_1(T, x) - u_2(T, x))^+ + \int_{t_0}^{T} \sup_{x \in T_{t_0,x_0}(s)} (f_1(s, x) - f_2(s, x))^- \, ds,$$

where $T_{t_0,x_0}(s)$ is set $\{x \in \Omega \mid x_{0i} - L(s - t_0) \leqslant x_i \leqslant x_{0i} + L(s - t_0),\ 1 \leqslant i \leqslant n\}$.

Uniqueness of generalized viscosity solutions is an immediate consequence of Theorem 1.1.

2. The Blow-Up Method

The proof of Theorem 1.1 is based on removing the dependence of H on t and x. We can accomplish this using the "blow-up" method. This method has also been used in Evans and Ishii [5].

Definition. Fix (t_0, x_0) in $(0, T) \times \Omega$. Let $\{\varepsilon_i\} \subset R^+$ and $\{x_i\} \subset \Omega$ be sequences such that $\varepsilon_i \downarrow 0$ and $x_i \to x_0$. Given the bounded Lipschitz function $u(t, x)$ on $[0, T] \times \Omega$, define the sequence of functions

$$u^i(t, x) = (u(t_0 + \varepsilon_i t, x_i + \varepsilon_i x) - u(t_0, x_i))/\varepsilon_i, \qquad i = 1, 2, \ldots.$$

Since u is Lipschitz on $[0, T] \times \Omega$, each function u^i is also Lipschitz on $[0, T] \times \Omega$. Then the sequence $\{u^i\}$ consists of functions which have the same Lipschitz constant and therefore, by Arzela–Ascoli, has a convergent subsequence in the sup norm to a Lipschitz function. If u^i denotes such a convergent subsequence, then the function

$$u^\infty(t, x) = \lim_{i \to \infty} u^i(t, x)$$

is called a *blow-up* of u (at (t_0, x_0)).

The point to be made in the next theorem is that blow-ups allow us to freeze the point (t_0, x_0). This effectively removes t_0, x_0 from the hamiltonian and the remaining hamiltonian depends continuously on p. Consequently, standard viscosity solutions apply.

Theorem 2.1. *Let (t_0, x_0) be fixed in $[0, T) \times \Omega$. Suppose that $\{t_0\} \times X \times P$ is a uniform Lebesgue set of $H(t, x, p)$ for any compact subsets X and P of Ω and also $\{t_0\} \times X$ is a uniform Lebesgue set of $f(t, x)$. Suppose that $u(t, x)$ is a generalized viscosity supersolution (resp. subsolution) of (1.1). With u^∞ a blow-up function of u (at (t_0, x_0)), it follows that u^∞ is a viscosity (in the ordinary Crandall–Lions sense) supersolution (resp. subsolution) of*

$$u_t(t, x) + H(t_0, x_0, D_x u(t, x)) = f(t_0, x_0) \qquad \text{in } [0, T) \times \Omega. \tag{2.1}$$

Remark. u^∞ is the limit of *any* convergent subsequence of $\{u^i\}$.

Proof. By definition, u^∞ is a viscosity supersolution of (2.1) if for any ϕ in, say $C^2([0, T] \times \Omega)$, if $(u^\infty - \phi)(t, x) \geqslant 0$ and at (τ_0, ξ_0) in $[0, T] \times \Omega$, $(u^\infty - \phi)(\tau_0, \xi_0) = 0$, then $\phi_t(\tau_0, \xi_0) + H(t_0, x_0, D_x\phi(\tau_0, \xi_0)) \leqslant f(t_0, x_0)$. Using Taylor's theorem, this is seen to be equivalent to the condition:

If

$$u^\infty(t, x) - \{u^\infty(\tau_0, \xi_0) + (t - \tau_0)\, \phi_t(\tau_0, \xi_0) + (x - \xi_0) \cdot D_x\phi(\tau_0, \xi_0) - K[(t - \tau_0)^2 + |x - \xi_0|^2]\} \geqslant k[(t - \tau_0)^2 + |x - \xi_0|^2] \tag{2.2}$$

for some positive constants K and k, then

$$\phi_t(\tau_0, \xi_0) + H(t_0, x_0, D_x\phi(\tau_0, \xi_0)) \leqslant f(t_0, x_0). \tag{2.3}$$

To establish this condition we define

$$\psi_i(t, x) \equiv \phi_t(\tau_0, \xi_0)(t - t_0 - \varepsilon_i\tau_0) + D_x\phi(\tau_0, \xi_0) \cdot (x - x_i - \varepsilon_i\xi_0) - K[(t - t_0 - \varepsilon_i\tau_0)^2 + |x - x_i - \varepsilon_i\tau_0|^2]/\varepsilon_i,$$

where $\varepsilon_i \downarrow 0$ and $x_i \to x_0$.

Let (τ_i, ξ_i), for each i, be points at which $u - \psi_i$ has a minimum.

Then, since u is Lipschitz continuous, we must have

$$|D\psi_i(\tau_i, \xi_i)| \leqslant C,$$

where C is a constant independent of i. Using the definition of ψ_i this gives

$$\left|\left\langle \phi_t(\tau_0, \xi_0) - \frac{2K}{\varepsilon_i}(\tau_i - t_0 - \varepsilon_i\tau_0), D_x\phi(\tau_0, \xi_0) - \frac{2K}{\varepsilon_i}(\xi_i - x_i - \varepsilon_i\xi_0)\right\rangle\right| \leqslant C$$

and so

$$|\langle \tau_i - t_0 - \varepsilon_i\tau_0, \xi_i - x_i - \varepsilon_i\xi_0 \rangle| \leqslant C\varepsilon_i$$

for another constant C independent of i. C will henceforth denote a generic constant independent of i. The last inequality allows us to conclude that

$$|\langle \tau_i - t_0, \xi_i - x_i \rangle| \leqslant C\varepsilon_i.$$

That is, $|\tau_i - t_0|/\varepsilon_i$ and $|\xi_i - x_i|/\varepsilon_i$ are both $o(1)$. Define

$$\psi(t, x) = \phi_t(\tau_0, \xi_0)(t - \tau_0) + D_x\phi(\tau_0, \xi_0)(x - \xi_0) - K[(t - \tau_0)^2 + |x - \xi_0|^2].$$

Note that

$$(u-\psi_i)^i\,(t,x) \equiv \varepsilon_i^{-1}[(u-\psi_i)(t_0+\varepsilon_i t, x_i+\varepsilon_i x)-(u-\psi_i)(t_0,x_i)]$$
$$= u^i(t,x)-\psi(t,x)+C(\tau_0,\xi_0)$$

where

C is a constant depending on (τ_0, ξ_0):

$$C(\tau_0,\xi_0)=\phi_t(\tau_0,\xi_0)\,\tau_0+D_x\phi(\tau_0,\xi_0)\,\xi_0+K(\tau_0^2+|\xi_0|^2).$$

From the definitions of u^i, ψ_i and ψ we have

$$\begin{aligned}(u-\psi_i)(\tau_i,\xi_i) &= u(\tau_i,\xi_i)\\ &\quad -(\phi_t(\tau_0,\xi_0), D_x\phi(\tau_0,\xi_0))\cdot(\tau_i-t_0-\varepsilon_i\tau_0, \xi_i-x_i-\varepsilon_i\xi_0)\\ &\quad +K[(\tau_i-t_0-\varepsilon_i\tau_0)^2+|\xi_i-x_i-\varepsilon_i\xi_0|^2]/\varepsilon_i\end{aligned}$$

and

$$(u^i-\psi)\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)=(u-\psi_i)(\tau_i,\xi_i)-u(t_0,x_i)/\varepsilon_i.$$

Thus, if $u-\psi_i$ is minimized at τ_i, ξ_i then $u^i-\psi$ is minimized at

$$\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right).$$

For each i, define m_i by

$$(u^i-\psi)\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)=m_i.$$

Then we have with $\bar{\lambda}\equiv\langle\phi_t(\tau_0,\xi_0), D_x\phi(\tau_0,\xi_0)\rangle$,

$$u^i(t,x)-m_i-\bar{\lambda}\cdot(t-\tau_0, x-\xi_0)+K[(t-\tau_0)^2+|x-\xi_0|^2]\geqslant 0 \quad (2.4)$$

and

$$u^i\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)-m_i-\bar{\lambda}\cdot\left(\frac{\tau_i-t_0}{\varepsilon}-\tau_0,\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right)$$
$$+K\left[\left(\frac{\tau_i-t_0}{\varepsilon_i}-\tau_0\right)^2+\left|\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right|^2\right]=0$$

Manipulate the last equation as

$$
\begin{aligned}
0=&\left[u^i\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)-u^\infty\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)\right]\\
&+\left[u^\infty\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)-\left\{u^\infty(\tau_0,\xi_0)+\bar{\lambda}\cdot\left(\frac{\tau_i-t_0}{\varepsilon_i}-\tau_0,\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right)\right.\right.\\
&\qquad\left.\left.-K\left[\left(\frac{\tau_i-t_0}{\varepsilon_i}-\tau_0\right)^2+\left|\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right|^2\right]\right\}\right]\\
&+[u^\infty(\tau_0,\xi_0)-u^i(\tau_0,\xi_0)]+[u^i(\tau_0,\xi_0)-m_i].
\end{aligned}
$$

Hence, using the condition (2.2) we have

$$
\begin{aligned}
0\geqslant&\left[u^i\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)-u^\infty\left(\frac{\tau_i-t_0}{\varepsilon_i},\frac{\xi_i-x_i}{\varepsilon_i}\right)\right]\\
&+k\left[\left(\frac{\tau_i-t_0}{\varepsilon_i}-\tau_0\right)^2+\left|\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right|^2\right]\\
&+[u^\infty(\tau_0,\xi_0)-u^i(\tau_0,\xi_0)]+[u^i(\tau_0,\xi_0)-m_i].
\end{aligned}
$$

However, by (2.4) we have for each i at (τ_0, ξ_0),

$$u^i(\tau_0,\xi_0)-m_i\geqslant 0.$$

Consequently, since $\{((\tau_i-t_0)/\varepsilon_i, (\xi_i-x_i)/\varepsilon_i)\}$ is bounded and $u^i\to u^\infty$ we get

$$k\left[\left(\frac{\tau_i-t_0}{\varepsilon_i}-\tau_0\right)^2+\left|\frac{\xi_i-x_i}{\varepsilon_i}-\xi_0\right|^2\right]\leqslant o(1)$$

so that

$$|\tau_i-t_0-\varepsilon_i\tau_0|=o(\varepsilon_i)\qquad\text{and}\qquad|\xi_i-x_i-\varepsilon_i\xi_0|=o(\varepsilon_i).\tag{2.5}$$

Next, from the definition of ψ_i we have

$$\partial\psi_i(t,x)/\partial t=\phi_t(\tau_0,\xi_0)-\frac{2K}{\varepsilon_i}(t-t_0-\varepsilon_i\tau_0),$$

$$D_x\psi_i(t,x)=D_x\phi(\tau_0,\xi_0)-\frac{2K}{\varepsilon_i}(x-x_i-\varepsilon_i\xi_0).$$

And from (2.5), $\lim_{i\to\infty}\partial\psi_i(\tau_i,\xi_i)/\partial t=\phi_t(\tau_0,\xi_0)$, $\lim_{i\to\infty}D_x\psi_i(\tau_i,\xi_i)=D_x\phi(\tau_0,\xi_0)$.

Summarizing, we have $\tau_i \to t_0$, $\xi_i \to x_0$; $u - \psi_i$ attains a minimum at (τ_i, ξ_i) and condition (ii) of Definition (1.2) is obviously satisfied. Then, since u is a generalized viscosity supersolution, we have

$$\limsup_{i \to \infty} [\partial \psi_i(\tau_i, \xi_i)/\partial t + H(t_0, x_0, D_x \psi_i(\tau_i, \xi_i))] \leqslant f(t_0, x_0)$$

so that by the preceding

$$\phi_t(\tau_0, \xi_0) + H(t_0, x_0, D_x \phi(\tau_0, \xi_0)) \leqslant f(t_0, x_0).$$

We have shown that u^∞ is a viscosity supersolution of (2.1). This completes the proof since the proof for subsolutions is entirely similar.

The following lemma proves that uniform Lebesgue sets of H and f are plentiful.

LEMMA 2.2. *For a.e. t, $\forall$ compact subsets X, $P \subset \Omega$, $\{t\} \times X \times P$ is a uniform Lebesgue set of $H(t, x, p)$ and $\{t\} \times X$ is a uniform Lebesgue set of $f(t, x)$.*

Proof. The lemma immediately follows from the following general result: Given a function, $F(t, z)$, F: $[0, T] \times \Omega \to R^1$, Ω open subset of R^m with F Lebesgue measurable in t and uniformly continuous in z on compact subsets of Ω then, for a.e. t,

$$\lim_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |F(t+s, z) - F(t, z)| \, ds = 0$$

uniformly in z on compact subsets of Ω. To prove this result we define for fixed t

$$G^t(r, z) = \frac{1}{2r} \int_{-r}^{r} |F(t+s, z) - F(t, z)| \, ds.$$

Since F is uniformly continuous in z, we have

$$\begin{aligned} |G^t(r, z) - G^t(r, y)| &= \left| \frac{1}{2r} \int_{-r}^{r} |F(t+s, z) - F(t, z)| \, ds \right. \\ &\qquad \left. - \frac{1}{2r} \int_{-r}^{r} |F(t+s, y) - F(t, y)| \, ds \right| \\ &\leqslant \frac{1}{2r} \int_{-r}^{r} \big| |F(t+s, z) - F(t, z)| - |F(t+s, y) - F(t, y)| \big| \, ds \end{aligned}$$

$$\leqslant \frac{1}{2r} \int_{-r}^{r} |F(t+s, z) - F(t+s, y)| \, ds$$

$$+ \frac{1}{2r} \int_{-r}^{r} |F(t, y) - F(t, z)| \, ds$$

$$\leqslant \sigma(|z - y|),$$

where σ is a positive function independent of t decreasing to 0 with its argument.

We have then that $G^t(r, z)$ is uniformly continuous in z for each t. Further, since F is Lebesgue measurable in t, there is a countable dense set $\{z_i\} \subset \Omega$ such that

$$\lim_{r \to 0} G^t(r, z_i) = 0.$$

But then $\lim_{r \to 0} G^t(r, z) = 0$ for every $z \in \Omega$ since we showed G^t is uniformly continuous in z. Then we conclude that $G^t(r, z) \to 0$ uniformly as $r \to 0$ for z in a compact subset of Ω. ∎

Remark. By Lemma 2.2 for a.e. $t_0 \in [0, T)$, if u is a generalized viscosity solution of $u_t + H(t, x, D_x u) = f(t, x)$ then u^∞ is a viscosity solution of $w_t + H(t_0, x_0, D_x w) = f(t_0, x_0)$.

We are now prepared to present the proof of Theorem 1.1.

3. Proof of Theorem 1.1

Let $(t_0, x_0) \in [0, T) \times \Omega$ be a fixed point. Let τ be so that $t_0 < \tau \leqslant T$ and define ξ by

$$u_1(\tau, \xi) - u_2(\tau, \xi) = \sup_{x \in T_{t_0, x_0}(s)} [u_1(\tau, x) - u_2(\tau, x)].$$

Recall that $T_{t_0, x_0}(s) = \{x \in \Omega \mid x_{0i} - L(s - t_0) \leqslant x_i \leqslant x_{0i} + L(s - t_0), 1 \leqslant i \leqslant n\}$. Define $\gamma(s) = \sup_{x \in T_{t_0, x_0}(s)} [u_1(s, x) - u_2(s, x)]$, $t_0 \leqslant s \leqslant T$. Then let $\{t_i\}$ and $\{x_i\}$ be sequences such that, $x_i \in T_{t_0, x_0}(t_i)$, $t_i \uparrow \tau$ and $x_i \to \xi$ as $i \to \infty$ with $\gamma(t_i) = u_1(t_i, x_i) - u_2(t_i, x_i)$. For $j = 1, 2$, put

$$u_j^i(t, x) = \frac{1}{\tau - t_i} [u_j(\tau + (\tau - t_i) t, x_i + (\tau - t_i) x) - u_j(\tau, x_i)]$$

and $u_j^\infty(t, x)$ the limit as $i \to \infty$ of a convergent subsequence of $\{u_j^i(t, x)\}_i$. In other words, u_j^∞ is a blow up of u_j, $j = 1, 2$, at (τ, x_i). Then, by Theorem 2.1 we have that

(a) u_1^∞ is a viscosity subsolution of $w_t + H(\tau, \xi, D_x w) = f_1(\tau, \xi)$ on $(-\infty, 0] \times \Omega$,

(b) u_2^∞ is a viscosity supersolution of $w_t + H(\tau, \xi, D_x w) = f_2(\tau, \xi)$ on $(-\infty, 0] \times \Omega$.

Then, we also have

$$\begin{aligned} u_1^\infty(-1, 0) - u_2^\infty(-1, 0) &= \lim_{i \to \infty} \frac{1}{\tau - t_i} [u_1(t_i, x_i) - u_1(\tau, x_i) \\ &\quad - (u_2(t_i, x_i) - u_2(\tau, x_i))] \\ &= -\partial u_1(\tau, \xi)/\partial t + \partial u_2(\tau, \xi)/\partial t \\ &= -\gamma'(\tau). \end{aligned}$$

But

$$\begin{aligned} u_1^\infty(0, x) - u_2^\infty(0, x) &= \lim_{i \to \infty} \frac{1}{\tau - t_i} \{[u_1(\tau, x_i + (\tau - t_i)\, x) - u_1(\tau, x_i)] \\ &\quad - [u_2(\tau, x_i + (\tau - t_i)\, x) - u_2(\tau, x_i)]\} \\ &= \lim_{i \to \infty} \frac{1}{\tau - t_i} \{[u_1(\tau, x_i + (\tau - t_i) x) - u_2(\tau, x_i + (\tau - t_i) x)] \\ &\quad - [u_1(\tau, x_i) - u_2(\tau, x_i)]\}. \end{aligned}$$

And, by definition of x_i, each term in this last limit is nonpositive. Consequently, combining the above pieces we may apply the comparison principle of Crandall and Lions [3] to get

$$-\gamma'(\tau) = u_1^\infty(-1, 0) - u_2^\infty(-1, 0) \leqslant [f_1(\tau, \xi) - f_2(\tau, \xi)]^-$$

and so $-\gamma'(\tau) \leqslant \sup_{x \in T_{t_0, x_0}(\tau)} [f_1(\tau, x) - f_2(\tau, x)]^-$ for $t_0 \leqslant \tau \leqslant T$. Integrating this inequality from t_0 to T we get

$$\gamma(t_0) \leqslant \gamma(T) + \int_{t_0}^{T} \sup_{x \in T_{t_0, x_0}(\tau)} [f_1(\tau, x) - f_2(\tau, x)]^- \, d\tau.$$

But $\gamma(t_0) = u_1(t_0, x_0) - u_2(t_0, x_0)$ and $\gamma(T) \leqslant \sup_{x \in \Omega} [u_1(x, T) - u_2(x, T)]^+$ and the theorem follows.

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